CHE 657

Process Analysis and Modeling II

Exercise Problems

8^{th} Edition

Prepared by

Dr. Hong-ming Ku

King Mongkut's University of Technology Thonburi
Chemical Engineering Department
Chemical Engineering Practice School

1. *Solving a Constrained Optimization Problem by Graphical Method, I*

Consider the following nonlinear constrained optimization problem:

$$f(x_1, x_2) = x_1^2 + 4x_2^2$$

$$\begin{aligned} s.t. \quad & x_1 + x_2 \geq 2 \\ & 0 \leq x_1 \leq 4 \\ & x_2 \leq 2 \end{aligned}$$

(a) Determine the maximum of the problem by graphical method.

(b) Determine the minimum of the problem by graphical method. Note that although you must show your solution by graph, you may have to use algebra or calculus to determine the exact answer.

2. *Solving a Constrained Optimization Problem by Graphical Method, II*

(a)Consider the following nonlinear constrained optimization problem:

$$f(x_1, x_2) = 10(x_1 - 3.5)^2 + 20(x_2 - 4)^2$$

$$\begin{aligned} s.t. \quad & x_1 + x_2 \leq 6 \\ & x_1 - x_2 \leq 1 \\ & 2x_1 + x_2 \geq 6 \\ & 0.5x_1 - x_2 \geq -4 \\ & x_1 \geq 1 \quad x_2 \geq 0 \end{aligned}$$

Determine the minimum of the problem by graphical method.

(b) Consider the following nonlinear constrained optimization problem:

$$f(x, y) = -x + 2y$$

$$\begin{aligned} s.t. \quad & x^2 + y^2 \leq 5 \\ & x + y \geq -1 \end{aligned}$$

Determine the minimum of the problem by graphical method.

<u>Instructions</u>: For Problems 3 through 20, classify each problem as either linear (LP), quadratic (QP), nonlinear (NLP), integer linear (ILP), or mixed integer linear (MILP) programming problem. If the problem is quadratic or nonlinear, state whether it is constrained. Do not solve each problem unless you are explicitly told to do so.

3. *Optimal Height and Diameter of an Absorption Tower*

An absorption tower containing wooden grids is to be used for absorbing SO_2 in a sodium sulfite solution. A mixture of air and SO_2 will enter the tower at a rate of 70,000 ft^3/min, temperature of 250°F, and pressure of 1.1 atm. The concentration of SO_2 in the entering gas is specified, and a given fraction of the entering SO_2 must be removed in the absorption tower. The molecular weight of the entering gas mixture may be assumed to be 29.1. Under the specified design conditions, the number of transfer units necessary varies with the superficial gas velocity as follows:

$$\text{Number of transfer units} = 0.32 G_s^{0.18}$$

where G_s is the entering gas velocity as lbm/(hr-ft^2) based on the cross-sectional area of the empty tower. The height of a transfer unit is constant at 15 ft. The cost for the installed tower is $1 per cubic foot of inside volume, and annual fixed charges amount to 20 percent of the initial cost. Variable operating charges for the absorbent, blower power, and pumping power are represented by the following equation:

$$\text{Total variable operating cost as \$/hr} = 1.8 \times 10^{-8} G_s^2 + \frac{81}{G_s} + \frac{4.8}{G_s^{0.8}}$$

The unit is to operate 8,000 hr/year. Formulate this problem as an optimization problem to minimize the annual cost and classify it.

You must simplify all your equations so that in the end the only variable(s) left in the formulation are the unknowns (decision variables).

4. *Formulation of a Manufacturing Optimization Problem*

A chemical manufacturing firm has discontinued production of a certain unprofitable product line. This has created considerable excess production capacity on the three existing batch production facilities. Management is considering devoting this excess capacity to one or more of three new products: call them products 1, 2, and 3. The available capacity on the existing units which might limit output is summarized in the following table, and each of the three new products requires the following processing time for completion:

Unit	**Available Time,** hours/week	**Productivity,** hours/batch		
		Product 1	**Product 2**	**Product 3**
A	20	0.8	0.2	0.3
B	10	0.4	0.3	…..
C	5	0.2	…..	0.1

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 batches per week. The profit per batch would be \$20, \$6, and \$8, respectively, on products 1, 2, and 3.

Formulate this problem as an optimization problem and classify it. Assume that the number of batches needs not be integer. Propose three feasible solutions (for example, one solution may involve the production of just product 1 and no product 2 and 3). Compare the objective function of your three feasible solutions.

5. *Maximizing Total Profit in Making Two Display Boxes*

A cabinet maker is asked by a jeweler to build two display boxes with lids made from a special shatter and break-proof transparent material. Only 8 square feet of the material are in stock at present and no more can be procured within the time available to build the boxes. For Box 1, the length has to be 1 foot whereas the width and height are to be equal to an unspecified size x_1. For Box 2, the height has to be 1 foot whereas the width and length are to be equal to unspecified size x_2.

The cost of the material and labor is \$ 0.5 per square foot of the outside of each box (Hint: There are 6 sides to each box!). The cabinet maker charged the jeweler proportional to the sum of the three dimensions of the boxes, i.e., width + length + height at the rate of \$ 2.00 for box 1 and \$ 2.50 per foot for Box 2. The cabinet maker's problem is to determine nonnegative values of x_1 and x_2 which will satisfy the material constraint on the lids of the two boxes and at the same time maximize his total profit. Formulate the cabinet maker's problem as an optimization problem and classify it.

6. *Formulation of a Manufacturing Optimization Problem*

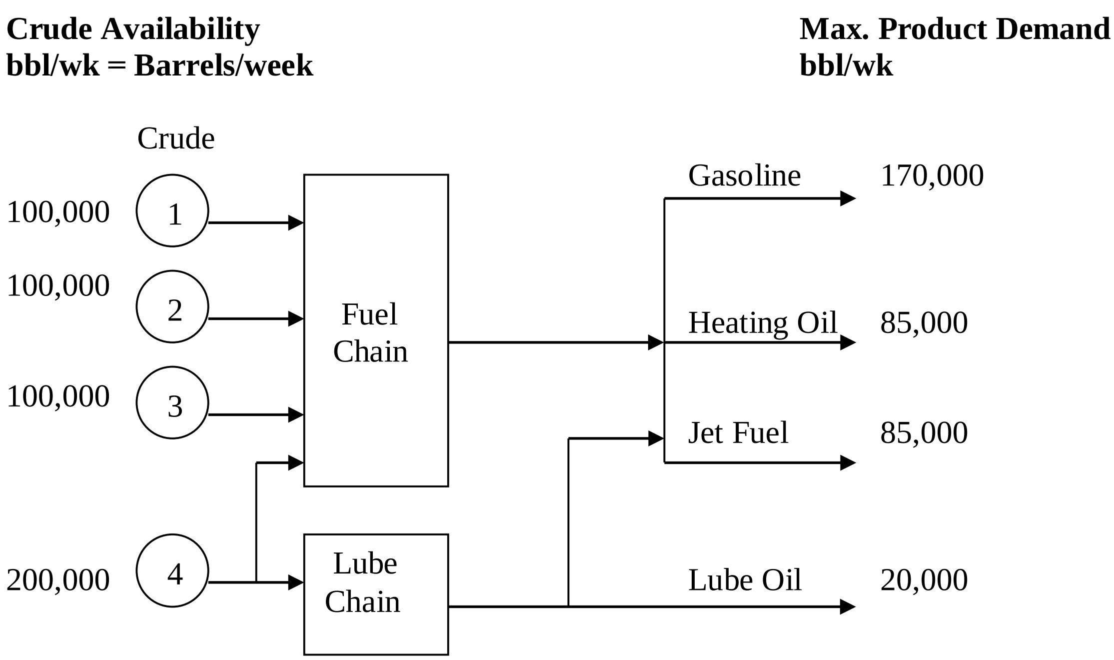
A certain plant can manufacture 5 different products in any combination. Each product requires time on each of three machines in the following manner (all numbers are in minutes/unit):

Product	Machine 1	Machine 2	Machine 3
A	12	8	5
B	7	9	10
C	8	4	7
D	10	0	3
E	7	11	2

Each machine is available 128 hours per week. Products *A*, *B*, and *C* are purely competitive, and any amounts made may be sold at respective prices of \$5, \$4, and \$5. The first 20 units of *D* and *E* produced per week can be sold at \$4 each, but all made in excess of 20 can only be sold at \$3 each. Variable labor costs are \$4 per hour for machines 1 and 2 and \$3 per hour for machine 3. Material costs are \$2 for products *A* and *C*, and \$1 for products *B*, *D*, and *E*. You wish to maximize profit to the firm. Formulate this problem as an optimization problem and classify it.

7. *Optimization of a Refinery Production*

A refinery has four different crudes which are to be processed to yield four products: gasoline, heating oil, jet fuel, and lube oil. There are maximum limits both on product demand (what can be sold) and crude availability. A schematic of the processing operation is as follows:



(a) Given the tabulated profits, costs, and yields in Table 1, formulate the problem as an optimization problem that maximizes the profit and classify it.

(b) Propose 3 feasible solutions (which are not necessarily optimal) to this problem.

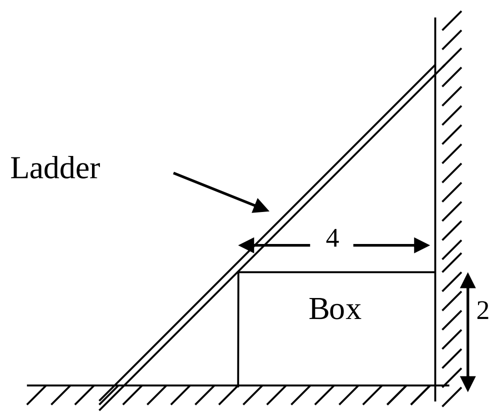
Table 1: Profits, Costs, and Yields Data

		Crude Type 1	2	3	4 Fuel Process	4 Lube Process	Product Value $/bbl
Yields	Gasoline	0.6	0.5	0.3	0.4	0.4	45.00
(bbl product	Heating Oil	0.2	0.2	0.3	0.3	0.1	30.00
per bbl crude)	Jet Fuel	0.1	0.2	0.3	0.2	0.2	15.00
	Lube Oil	0	0	0	0	0.2	60.00
	Other*	0.1	0.1	0.1	0.1	0.1	
Crude cost $/bbl		15.00	15.00	15.00	25.00	25.00	
Operating cost $/bbl		5.00	8.50	7.50	3.00	2.50	

* "Other" refers to losses in processing

8. *Finding the Shortest Length of a Ladder*

A rectangular box of height 2 meters and width 4 meters is placed adjacent to a wall (see figure below). Formulate this problem as an optimization and classify it. Find the length of the shortest ladder that can be made to lean against the wall.



9. *Formulation and Minimization of a Heat Transfer Problem*

It is desired to cool a gas [C_P = 0.3 Btu/lbm-°F] from 195 to 90°F, using cooling water [C_P = 1.0 Btu/lbm-°F, ρ = 62.4 lbm/ft^3] at 80°F in a counter-current heat exchanger. Water costs $0.20/1000 ft^3, and the annual fixed charges for the exchanger are $0.50/ft^2 of heat transfer area. The heat transfer coefficient is U = 8 Btu/hr-ft^2-°F for a gas rate of 3000 lbm/hr. The heat exchanger is operated 365 days/year (24 hours/day).

Formulate this heat exchange operation as an optimization problem to minimize the total annual cost and classify it.

Hint: ΔT_{lm} = log-mean $\Delta T = (\Delta T_2 - \Delta T_1)/\ln(\Delta T_2/\Delta T_1)$
where ΔT_2 and ΔT_1 are the terminal temperature differences of heat exchanger.

10. *Optimum Thickness of Insulation for a Furnace*

To reduce heat losses, the exterior flat wall of a furnace is to be insulated. To determine the optimum insulation thickness, it is necessary to consider and balance the costs of the insulation and the value of the energy saved by adding the insulation. The rate of heat transfer Q through the wall is:

$$Q = UA\,(T_{\text{furnace}} - T_{\text{wall}})$$

where T is in °F and Q is in Btu/hr. The overall heat transfer coefficient U is related to the outside convective heat transfer coefficient h and the thermal conductivity of insulation k by:

$$\frac{1}{U} = \frac{1}{h} + \frac{t}{12k}$$

where t is the thickness in inches of the insulation.

We wish to maximize the savings in total operating cost, savings expressed as the difference between the dollar value of the heat conserved minus the cost of the insulation over a span of 5 years (after that, the insulation will have to be replaced). Formulate this problem as an optimization problem, classify it, and obtain the optimum value of t (in inches) using the following data:

Temperature inside the furnace	500°F (constant)
Air temperature outside wall	Assume constant at 70°F
Heat transfer coefficients	
Outside air film h	4.0 Btu/(hr)(ft^2)(°F)
Conductivity of insulation k	0.03 Btu/(hr)(ft)(°F)
Total cost of insulation (per unit area per inch of thickness)	\$0.75/(ft^2)(per inch of thickness)
Values of energy saved (i.e. the dollar difference between adding insulation and having no insulation for every 1 million Btu is \$0.60.)	\$0.60/10^6 Btu
Hours of operation	8700 hours/year

Prove that your optimum t is a maximum. **Hint:** The heat transfer area A is constant.

11. *An Airline Assignment Problem*

Large airlines tend to base their route structure around the hub concept. An airline will try to have a large number of flights arrive at the hub airport during a certain short interval of time, e.g. 9 A.M. to 10 A.M. and then have a large number of flights depart the hub shortly thereafter, e.g. 10 A.M. to 11 A.M. This allows customers of that airline to travel between a large combination of origin/destination cities with one stop and at most one change of planes. For example, United Airlines uses Chicago as a hub, Delta Airlines uses Atlanta, TWA uses St. Louis, and American uses Dallas/Fort Worth. A desirable goal in using a hub structure is to minimize the amount of changing of planes (and the resulting moving of baggage) at the hub. The following problem illustrates this airline assignment problem.

A certain airline has 6 flights arriving at O'Hare airport between 9 and 9:30 A.M. The same 6 airplanes depart on different flights between 9:40 and 10:20 A.M. The average numbers of people transferring between incoming and leaving flights appear in the table below:

	L01	L02	L03	L04	L05	L06
I01	20	15	16	5	4	7
I02	17	15	33	12	8	6
I03	9	12	18	16	30	13
I04	12	8	11	27	19	14
I05	0	7	10	21	10	32
I06	0	0	0	6	11	13

Flights I05 arrives too late to connect with L01. Similarly, I06 is too late for flights L01, L02, and L03.

All the planes are identical. A decision problem is which incoming flight should be assigned to which outgoing flight. For example, if incoming flight 102 is assigned to leaving flight L03, 33 people (and their baggage) will be to remain on their plane at the stop at O'Hare. Formulate this assignment problem as an optimization and classify it.

12. *Toll-Way Staffing Problem*

The Northeast Tollway out of Chicago has a toll plaza with the following staffing demands during each 24-hour period:

Hours	Collectors Needed
12 a.m. to 6 a.m.	2
6 a.m. to 10 a.m.	8
10 a.m. to 12 a.m.	4
12 a.m. to 4 p.m.	3
4 p.m. to 6 p.m.	6
6 p.m. to 10 p.m.	5
10 p.m. to 12 Midnight	3

Each collector works 4 hours, is off one hour, and then works another 4-hours. A collector can be started at any hour. Assuming the objective is to minimize the number of collectors hired, formulate this problem as an optimization problem.

13. *Formulation of a Manufacturing Problem*

A chemical company manufactures four products (1, 2, 3, and 4) on two machines (*X* and *Y*). The time (in minutes) to process one unit of each product on each machine is shown below:

		Machine	
		X	*Y*
Product	1	10	27
	2	12	19
	3	13	33
	4	8	23

The profit per unit for each product (1, 2, 3, and 4) is $10, $12, $17, and $8 respectively. Each product can be produced on either machine.

The factory is very small and this means that floor space is very limited. Only one week's production is stored in 50 square meters of floor space where the floor space taken up by each product is 0.1, 0.15, 0.5, and 0.05 (square meters) for products 1, 2, 3, and 4 respectively.

Customer requirements mean that the amount of product 3 produced should be related to the amount of product 2 produced. Over a week, twice as many units of product 2 should be produced as product 3.

Machine *X* is out of action (for maintenance/because of breakdown) 5% of the time and machine *Y* 7% of the time.

Assuming a working week 35 hours long, formulate the problem of how to manufacture these products as a linear program so as to maximize the total weekly profit and classify it.

14. *Maximizing the Monetary Value of a Cargo Load*

A cargo load is to be prepared from five types of articles. The weight w_i, volume v_i, and monetary value c_i of different types of articles are given below.

Article Type	w_i	v_i	c_i
1	4	9	5
2	8	7	6
3	2	4	3
4	5	3	2
5	3	8	8

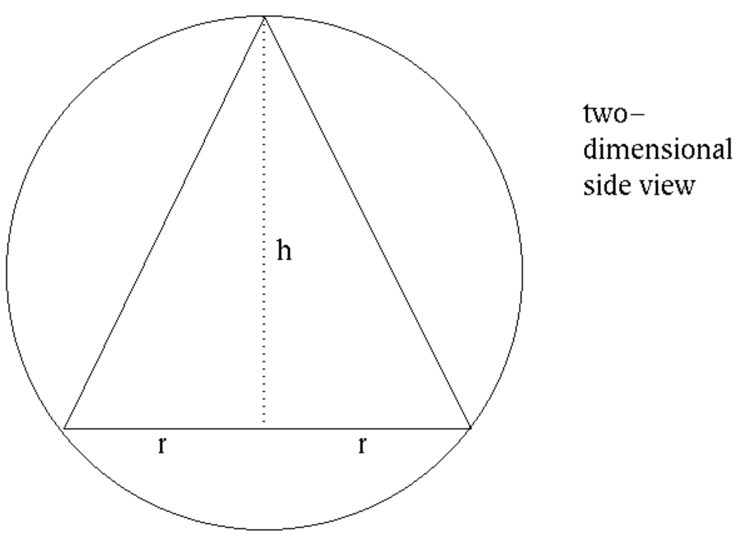
Formulate this optimization problem which will determine the number of articles x_i selected from the i^{th} type (i = 1, 2, 3, 4, 5), so that the total monetary value of the cargo load is maximized. The total weight and volume of the cargo cannot exceed the limits of 2000 and 2500 units, respectively. Classify the problem but do not solve for the optimum. Note that it is not possible to load a fraction of an article.

15. *Formulating and Solving a Vessel-Design Optimization Problem*

As a starting chemical engineer at Vessels Fabrication Co., Ltd., your first assignment is to design a cone-shaped vessel. Specifically, your job is to calculate the dimensions (radius r and height h) of the cone with a maximum volume that can be inscribed (fit into) in a sphere of radius 2 meters. Formulate this assignment as an optimization problem, classify it, and solve it. Note that the volume of a cone is given by

$$V = \frac{1}{3}\pi\ r^2 h$$

Also, show that the volume found is indeed a maximum. A two-dimensional side view of the geometry is shown below.



16. ***Formulation of a Cargo Plane Optimization Problem***

A cargo plane has three compartments for storing cargo: front, center, and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tons)	Space capacity (m^3)
Front	10	6800
Center	16	8700
Rear	8	5300

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane. The following four cargoes are available for shipment on the next flight:

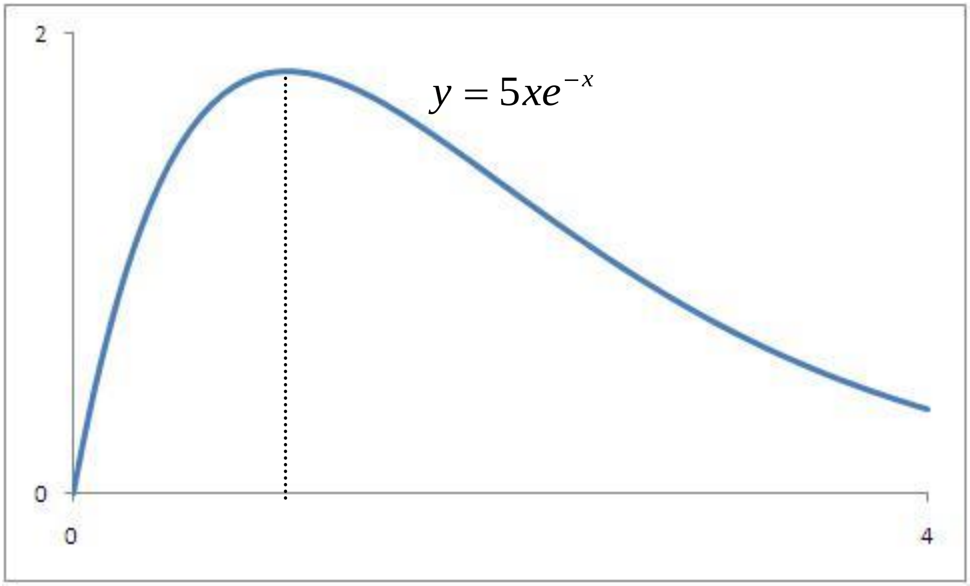
Cargo	Weight (tons)	Volume (m^3/ton)	Profit ($/ton)
$C1$	18	480	310
$C2$	15	650	380
$C3$	23	580	350
$C4$	12	390	285

Any proportion of these cargoes can be accepted. The objective is to determine *how much* (if any) of each cargo $C1$, $C2$, $C3$ and $C4$ should be accepted and *how to distribute* each among the compartments so that the total profit for the flight is maximized. Formulate the above optimization problem and classify it, but do not solve it.

17. ***Minimum Area under a Curve***

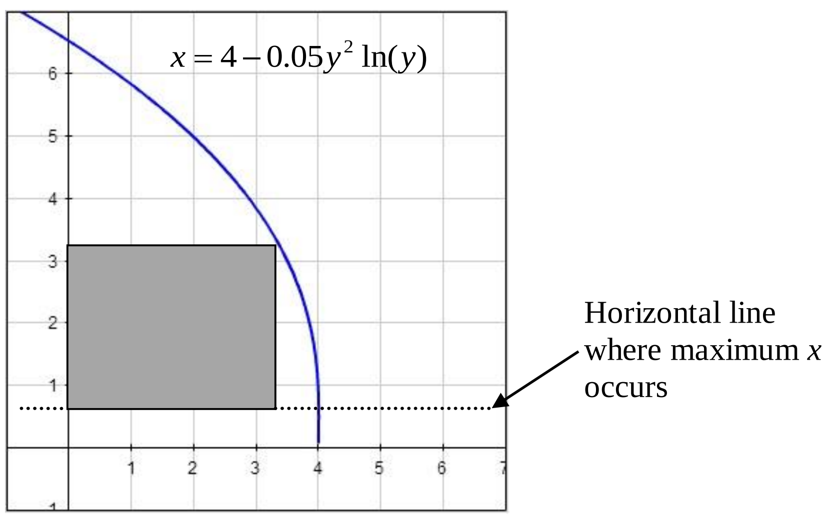
The graph below shows the function $\boldsymbol{y = 5xe^{-x}}$ over the domain [0, 4] with a maximum. A shaded rectangle is enclosed under the curve between the curve, the x-axis, and the vertical line where the maximum occurs, as shown. Calculate the minimum **unshaded** area which is possible under the curve between the domain x = [0, 4] by formulating it as an optimization problem, and classify it. Note that this is equivalent to maximizing the area of the shaded rectangle.

Recall the formula for integration by parts: $$\int u\,dv = uv - \int v\,du,$$



18. ***Solving an Optimization Problem by Calculus***

The graph below shows the function $x = 4 - 0.05y^2 \ln(y)$. A shaded rectangle is enclosed inside the curve bounded by the y-axis and the horizontal line where x has the largest value. Calculate the minimum value of the **unshaded** area inside the curve bounded by the y-axis and this horizontal line when one maximizes the area of the shaded rectangle. Carry 3 decimal places in all your calculations.



19. *Formulating a Compressor Storage Problem*

Acme Inc. engages in the business of building large and expensive compressors. The company operates 3 plants, called Plant *A*, *B*, and *C*, in different locations which have different manufacturing capacities (number of compressors built at the end of each production week). Because of their sizes, all the compressors built at each plant must be transported to a warehouse for storage at the end of each production week. The company owns three warehouses in different locations, which also have different storage capacities. The total cost associated with storing one compressor (e.g. transportation cost, space rental fee, etc.) at each warehouse is known and given in the table below.

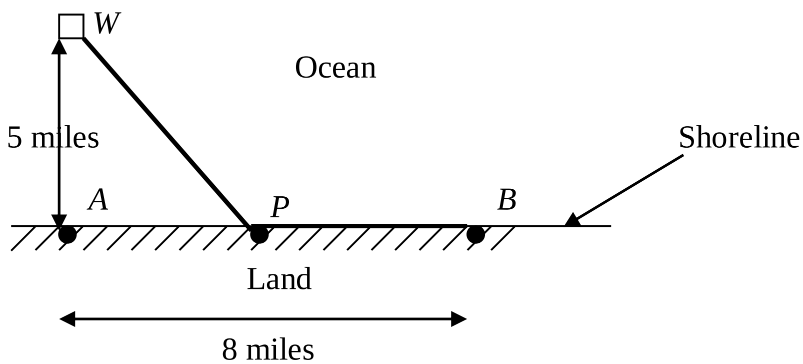
As a chemical engineer working for Acme, you are asked to formulate this optimization problem and classify it. The objective is to minimize the total cost of storing all compressors manufactured at the end of a production week at the 3 warehouses.

Cost of Storing a Compressor from a Plant at a Warehouse

	Warehouse 1	**Warehouse 2**	**Warehouse 3**	**Plant Capacity**
Plant *A*	15	20	50	30
Plant *B*	15	30	40	25
Plant *C*	30	10	10	40
Max. Storage Capacity	40	30	35	

20. *Minimizing the Piping Cost for an Offshore Oil Well*

An offshore oil well is located in the ocean at a point *W*, which is 5 miles from the closest shorepoint *A* on a straight shoreline (see the figure below). The oil is to be piped to a shorepoint *B* that is 8 miles from *A* by piping it on a straight line under water from *W* to some shorepoint *P* between *A* and *B* and then on to *B* via a pipe along the shoreline. If the cost of laying pipe is $100,000 per mile under water and $75,000 per mile over land, formulate this piping problem as an optimization problem to minimize the cost of laying the pipe, classify it as either unconstrained NLP, constrained NLP, LP, MILP, or MINLP, and determine analytically where the point *P* should be located that gives the least possible cost? Compute this minimum cost as well.



21. *Characterization of Stationary Points*

Identify and characterize (as local/global minimum, local/global maximum, or saddle point) the stationary points of the following functions. Compute the value of the function at each stationary point as well.

(a) $f(x) = \dfrac{x^2}{(x+1)(x-2)}$

(b) $f(x) = (x-1)^2 \exp[x(x+3)]$

(c) $f(x_1, x_2) = \ln(x_1 x_2) + x_1 x_2 - x_2^2 - 2x_1$

(d) $f(x_1, x_2) = 2x_1^3 + 4x_1 x_2^2 - 10x_1 x_2 + x_2^2$

(e) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 - x_2^2 x_3 + \ln(x_3) - 4x_3 + 4$

(f) $f(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 + x_2^2$

(g) $f(x_1, x_2) = x_1 x_2 - 4\sin(x_1) + \ln\left(\dfrac{x_1}{x_2}\right) \qquad 0 \le x_1, x_2 \le \pi$

(h) $f(x_1, x_2, x_3) = \exp(-x_1^2) + 2x_1^2 + x_2^2 + x_2 x_3 + x_3^3$

(i) $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

(j) $f(x_1, x_2, x_3) = \dfrac{x_1}{(1-x_2)} - \exp(x_2) + 2x_1^3 - 4x_1 + 2x_2 + x_1 x_2 x_3$

(k) $f(x, y) = 3x - x^3 - 3xy^2$

(l) $f(x, y) = x^4 + 2x^2 y - x^2 - y^2$

(m) $f(x_1, x_2) = (x_1 - 2)\exp(x_1 + x_2^2) + x_2$

(n) $f(x_1, x_2) = 2x_1^2 + x_1^2 x_2 - x_1 x_2^2 + 2x_1 x_2 + 8x_1$

22. *Solving a Heat Transfer Minimization Problem*

Refer to Problem 9, which is a problem to minimize the total annual cost of a heat exchanger.

(a) Solve for the optimum analytically.

(b) Use the function *fminunc* in MATLAB to solve for the minimum annual cost. Briefly explain the application of this function.

23. *Optimization of a Solvent-Extraction Operation*

A solvent-extraction operation is carried out continuously in a plate column with gravity flow. The unit is operated 24 hr/day. A feed rate of 1500 ft^3/day must be handled 300 days per year. The allowable velocity per square foot of cross-sectional tower area is 40 ft^3 of combined solvent and charge per hour. The annual fixed costs for the installation can be predicted from the following equation:

$$C_F \;=\; 6{,}800F^2_{SF} \;-\; 48{,}000F_{SF} \;+\; 120{,}000 \quad \text{\$/year}$$

where F_{SF} = cubic feet of solvent per cubic foot of feed. Operating and other variable costs depend on the amount of solvent that must be recovered, and those costs are $0.04 for each cubic foot of solvent passing through the tower.

(a) What tower diameter should be used for optimum conditions of minimum total cost per year?

c) What is the answer in Part (a) if the annual fixed costs for the installation follow the equation below:

$$C_F \;=\; 400F^4_{SF} \;-\; 1{,}800F^3_{SF} \;-\; 12{,}000F_{SF} \;+\; 120{,}000 \quad \text{\$/year}$$

Write a MATLAB program to implement the following methods to find the optimum diameter D_{opt}, correct to 3 decimal places, i.e. $|D_{opt,k+1}| - |D_{opt,k}| \geq 0.001$.

(i) Newton-Raphson method
(ii) Quasi-Newton method using a step size of $h = 0.1$
(iii) Secant method

24. *Single-Variable Search Using 3 Different Techniques*

Carry out a single-variable search of the function

$$f(x) \;=\; x \;+\; 4/x \qquad \text{on the interval } [1, 5]$$

using (a) Newton-Raphson method
(b) Quasi-Newton method using a step size of $h = 0.1$
(c) Secant method

Carry out 5 iterations in each method using Excel. Also determine the true minimum of $f(x)$ analytically and compare the accuracy of the minima estimated by these three methods.

25. *Solving for the Optimal Height and Diameter of an Absorption Tower*

Refer to Problem 3 again. Recall that G_s, the superficial gas velocity, is the decision variable to be minimized in the cost function. Given that the optimal value of G_s lies between 1000 and 2000 lbm/ft^2-hr, do the following:

(a) Use *fminbnd* function in MATLAB to solve for the optimal tower height, diameter, and minimum annual cost. What is the number of iterations, the number of function evaluations, and the algorithm used as reported by MATLAB?

(b) Write a MATLAB script file to implement Golden Section search to find the optimal G_s, accurate to 1 decimal place. What is the total number of iterations required?

26. *Solving for a Maximum Using Golden Section*

Find the maximum of the function xe^{-x} over the interval $0 \le x \le 4$ by performing 6 function evaluations with the Golden Section search by hand. What is the accuracy of the result after 6 iterations? Confirm analytically that the solution is in fact a local maximum. Prove that it is a global maximum.

27. *Steepest Descent with Golden Section*

Consider a chemical process for which there are 2 design variables x and y. After cost estimation and profitability analysis, it is determined that it is desirable to find the values of x and y that minimizes the cost function C:

$$C = 6.5x^{0.5} + \frac{3000}{xy} + 5y + 2x^2 + 60$$

The initial guess is $x^{(0)} = 2$ and $y^{(0)} = 7$.

In this problem you are asked to perform one iteration of the steepest descent method for solving this problem. That is, find $x^{(1)}$ and $y^{(1)}$, the next estimates of the optimum.

For the variable optimization with respect to the parameter λ (line search, which indicates how far to move in the steepest descent direction), use the Golden Section method. Assume C is unimodal with respect to λ, and that $0 \le \lambda \le 0.1$. You need to perform only 3 iterations of the Golden Section method (so you will have to evaluate C only 4 times). You must show all intermediate results in each iteration of Golden Section. Do not use Excel, MATLAB, or the programmable feature in your calculators and do not show your answers in the form of tables.

You should express your answer in terms of an interval. That is, determine what two numbers $x^{(1)}$ lies between and what two numbers $y^{(1)}$ lies between. Carry 4 decimal places in all your calculations.

28. *Unconstrained Optimization of a Two-Variable Problem*

Consider the following unconstrained optimization problem of two variables. We like to use the following strategy to minimize the given objective function.

$$\textit{Minimize}\ \ f(x) = x_2^3 - x_1 x_2 + x_1^2 - x_2$$

Iteration 1: Starting at $[x_1, x_2] = [1, 1]$, carry out one iteration of Steepest Descent. Note that you must minimize λ in the line search.

Iteration 2: Carry out one iteration of Conjugate Direction Search based on the search direction and the solution in Iteration 1. Fix the value of the second element of your conjugate vector to 1 and calculate its first element, i.e. $\boldsymbol{s} = [s_1, 1]^T$. You also must minimize λ in this iteration.

Iteration 3: Carry out one iteration of Newton's method based on the solution in Iteration 2.

Calculate $f(\boldsymbol{x})$ in each iteration and comment on the improvement of the objective value, if any, in each iteration. What are the absolute relative percentage deviations of x_1, x_2, and $f(\boldsymbol{x})$ after three iterations when compared to the global minimum in this problem? Carry 4 decimal places in all your calculations.

Formula: The inverse of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is* $\dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $\det(A) = ad - bc$

29. *Line Search with Golden Section in a 2-Variable Problem*

A one-dimensional search is to be carried out from the point [0, 3] using $\boldsymbol{s} = [1\ 1]^T$; $\boldsymbol{s}$ is the search direction on the function:

$$f(x) = (x_1 - 1)^2 + 4(x_2 - 2)^2$$

Use Golden Section method to find the minimum along this search direction (with respect to λ), with an accuracy of 4 decimal places. Also, find this optimum value analytically and show that it is (–0.6, 2.4).

30. *Line Search with Golden Section in a 2-Variable Problem*

A one-dimensional search is to be carried out from the point [1, 3] using $\boldsymbol{s} = [1\ -2]^T$; $\boldsymbol{s}$ is the search direction on the function:

$$f(\mathbf{x}) = 2x_1^3 + 4x_1x_2^2 - 10x_1x_2 + x_2^2$$

Implement Golden Section method in MATLAB to find the minimum along this search direction (with respect to λ), with an accuracy of 4 decimal places. Also, use *fminbnd* to find the optimum value of the step size λ.

31. *Conjugate Search Directions for a Quadratic Function*

The general equation in matrix form for a quadratic function can be expressed as:

$$f(\boldsymbol{x}) = a + b^T\boldsymbol{x} + \tfrac{1}{2}\,\boldsymbol{x}^{\mathbf{T}}C\boldsymbol{x}$$

where b^T is a ***1xn*** matrix and C is an ***nxn*** symmetric matrix, and n is the total number of variables.

(a) For the function

$$f(\boldsymbol{x}) = 5x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 - x_1x_3 - 4x_2 + 5x_3 + 8$$

show that the vectors $[1\ \ 0\ \ 0]^T$ and $[1\ \ 1\ \ 12]^T$ are conjugate directions with respect to the C matrix.

(b) Compare the C matrix with the Hessian matrix $H(\boldsymbol{x})$ of $f(\boldsymbol{x})$ and comment on their relationship.

(c) Is the function in Part (a) strictly convex, convex, strictly concave, concave, or none of the above?

32. *Combined Search Techniques in a Multivariable Minimization Problem*

Consider a chemical process for which there are 2 design variables x_1 and x_2. After cost estimation and profitability analysis, it is determined that it is desirable to find the values of x_1 and x_2 that <u>minimizes</u> the cost function $f(\boldsymbol{x})$:

$$f(x_1, x_2) = x_1^3 + x_1x_2^2 - x_1x_2$$

In searching for the minimum, perform 2 iterations using the following combination of search techniques:

i) Iteration 1: use Steepest Descent search
ii) Iteration 2: use Newton's method

Use an initial guess of $x_1 = 1$ and $x_2 = 1$. In Iteration 1, you must carry out a line search (i.e. minimize the step size λ completely), and in Iteration 2 use the solution from Iteration 1 as the starting point. Calculate and report the value of $f(\boldsymbol{x})$ in each iteration.

33. *Unconstrained Optimization of a Two-Variable Problem*

Consider the following unconstrained optimization problem of two variables. Carry out one iteration of each method below to minimize the given objective function.

$$\textit{Minimize} \;\; f(x) = x_1^4 + 2x_1x_2^2 - x_1x_2$$

Method 1: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Steepest Descent search. Note that you must minimize λ, whose optimal value is less than 0.2, in the line search accurate to 4 decimal places. You may choose any line-search (one-variable) optimization technique but a fast convergence method is recommended. You must show all work instead of using your programmable calculator to give the final answer.

Method 2: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Conjugate Direction search. This search direction must be conjugate to the steepest descent search direction in Method 1. Fix the value of the second element of your conjugate vector to 1 and calculate its first element, i.e. $\boldsymbol{s} = [s_1, 1]^T$. Once again, you must minimize λ, whose value is less than 0.2, in the line search accurate to 4 decimal places. You may choose any line-search (one-variable) optimization technique but a fast convergence method is recommended. You must show all work instead of using your programmable calculator to give the final answer.

Method 3: Starting at $[x_1, x_2]^T = [1, 0]^T$, carry out one iteration of Newton's method

Calculate $f(\boldsymbol{x})$ in each method after one iteration. What are the absolute relative percentage deviations of x_1, x_2, and $f(\boldsymbol{x})$ of the three methods when compared to the global minimum in this problem? Carry 4 decimal places in all your calculations.

<u>Formula</u>: The inverse of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\; is \;\; \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $\det(A) = ad - bc$

34. ***Using Steepest Descent to Solve a Multivariable Minimization Problem***

Consider an unconstrained optimization problem with two decision variables x and y. It is desirable to find the values of x and y that minimizes the objective function $f(x, y)$:

$$f(x,y) = (y - x^2)^2 + (5 - x)^2$$

Carry out one iteration of the Steepest Descent method. Use an initial guess of $x = 2$ and $y = 5$. In your line search, carry out 5 iterations of Golden-Section Search.

35. ***Conjugate-Gradient and Newton's Methods for a Multivariable Problem***

For the following quadratic function,

$$f(\boldsymbol{x}) = 4(x_1 - 5)^2 + (x_2 - 6)^2$$

starting at $\boldsymbol{x}^0 = [1 \;\; 1]^T$.

(a) Use the Fletcher-Reeves (conjugate-gradient) search to find the minimum.

(b) Use Newton's method to find the minimum.

36. ***Newton's Methods for Another Multivariable Problem***

Solve the following problem by Newton's method:

$$\text{Minimize} \quad f(\boldsymbol{x}) = 1 + x_1 + x_2 + x_3 + x_4 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$

starting from $\boldsymbol{x}^0 = [-3 \;\; -30 \;\; -4 \;\; -0.1]^T$.

37. ***Steepest Descent and Conjugate Direction Methods***

Consider the maximization of

$$f(\boldsymbol{x}) = 12x_1 - 2x_1^2 - 2x_2^2 + 2x_1x_2 - 12$$

(a) Find the stationary point analytically, and prove that it is a local maximum.

(b) Carry out 3 iterations of steepest descent (with the line search) from the point $[1 \;\; 1]^T$. What are the values of x_1, x_2, and $f(\boldsymbol{x})$ after 3 iterations?

(c) Determine the maximum using conjugate search directions. Since the number of variables is 2 and $f(\boldsymbol{x})$ is quadratic, it should take you only 2 iterations to find the maximum.

38. *Conjugate-Gradient Method, Newton's Method, and fminunc for a Multivariable Problem*

For the following optimization problem,

$$\text{Minimize } f(\boldsymbol{x}) \;=\; \exp(-x_1^2) + 2x_1^2 + x_2^2 + x_2x_3 + x_3^3$$

Write a MATLAB program to implement the following:

(a) 10 iterations of the Fletcher-Reeves (conjugate-gradient) search to find the minimum starting at $\boldsymbol{x}^0 = [0\; -0.5\;\; 0.5]^T$. Do not perform the line search in each iteration. Instead, use a fixed step size of $\lambda = 0.1$.

(b) 5 iterations of Newton's method to find the minimum starting at $\boldsymbol{x}^0 = [0\;\; -1\;\; 1]^T$.

(c) Use *fminunc* function to solve for the minimum at $\boldsymbol{x}^0 = [1\;\; 1\;\; 1]^T$. Do not use the default algorithm of Quasi-Newton line search because it will not converge the problem at the given starting point. Instead, use the trust-region Newton algorithm.

39. *Sequential Simplex Method and MATLAB's fminsearch*

For the following function

$$f(\boldsymbol{x}) \;=\; [\exp(x_1)][4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1]$$

(a) Carry out 5 steps of sequential simplex method to find the minimum, using a simplex with a length of 0.2 and a starting point of $\boldsymbol{x}^0 = [2\; -2]^T$.

(b) Use *fminsearch* in MATLAB to find the minimum starting at $\boldsymbol{x}^0 = [2\; -2]^T$.

40. *Combined Search Techniques in a Multivariable Minimization Problem*

Consider a chemical process for which there are 2 design variables x_1 and x_2. After cost estimation and profitability analysis, it is determined that it is desirable to find the values of x_1 and x_2 that <u>minimizes</u> the cost function $f(\boldsymbol{x})$:

$$f(x_1, x_2) = x_1^2 + 2x_1x_2^2 - x_1x_2$$

In searching for the minimum, perform 3 iterations using the following combination of search techniques:

Iteration 1: use Steepest Descent search
Iteration 2: use a search direction which is conjugate to the search direction in Iteration 1
Iteration 3: use Newton's method

Use an initial guess of $x_1 = 2$ and $x_2 = 1$. In Iteration 1 and 2, you must carry out a line search (i.e. minimize the step size λ completely), and in each iteration use the solution from the previous iteration as the starting point. Calculate and report the value of $f(\boldsymbol{x})$ in each iteration as well.

41. *Profit Maximization in a Chemical Process as an LP*

A chemical company produces 2 kinds of product, namely Saccharin and Nutrasweet which are sugar substitutes because of their extreme sweetness. Saccharin generates a profit of $4100 per batch, requires 4 hours of production per batch in Plant 1 and 2 hours per batch in Plant 2. Nutrasweet has a profit contribution of $5900 per batch, requires 2 hours per batch in Plant 1 and 3 hours per batch in Plant 2 to produce. The available time in one production cycle in Plant 1 and Plant 2 are 160 hours and 180 hours, respectively. Each hour used in Plant 1 costs $90, while each hour spent in Plant 2 costs $60.

(a) Formulate the problem as an LP in order to maximize the profit in one production cycle. Solve the LP problem graphically.

(b) Solve the same problem again using the Simplex algorithm.

42. *Formulation of an LP Problem*

Anti-H5N1 Inc. is a pharmaceutical company specialized in producing an expensive vaccine to inoculate humans against the avian flu. The vaccines come in two varieties, one for immunization against Strain A (called AFVAX-A) and one against Strain B (called AFVAX-B). A facility is used to produce both AFVAX-A and AFVAX-B. Because of the high demands for its products, the company must produce a total of at least 14 batches in one production cycle, of which at least 3 batches must be AFVAX-A and at least 2 batches must be AFVAX-B. It takes 8.0 hours to produce one batch of the Strain-A vaccine and 6.25 hours to produce one batch of the Strain-B vaccine. A maximum of 100 operation hours is available at the facility in one production cycle. The net profit for the Strain-A vaccine is $25,000 per batch, while that of the Strain-B is $19,800 per batch.

Anti-H5N1 Inc. wants to know how much of AFVAX-A and AFVAX-B they must produce in one production cycle in order to achieve the highest possible profit.

(a) Formulate an LP model for this problem. Treat all variables as continuous and use the graphical method to solve this model.

(b) Now treat the model as an ILP problem, i.e. the number of batches must be an integer. Use the graphical method again to solve for the optimum. You must show all your work to get full credit.

43. *Optimal Manufacturing of Candies*

A confectioner manufactures 2 kinds of candy bars: Ergies (packed with energy for kiddies) and Nergies (the "low-calorie" nugget for weight watchers without will power). Ergies sell at a profit of 50 cents per box while Nergies have a profit of 60 cents per box. The candy is processed in 3 main operations: blending, cooking, and packaging. The following table records the average time in *minutes* required by each box of candy, for each of the three activities.

	Blending	**Cooking**	**Packaging**
Ergies	1	5	3
Nergies	2	4	1

During each production run, the blending equipment is available for a maximum of 14 machine hours, the cooking equipment for at most 40 machine hours, and the packaging equipment for at most 15 machine hours.

(a) If each machine can be allocated to the making of either type of candy at all times that it is available for production, determine how many boxes of each kind of candy the confectioner should make in order to realize the maximum profit. Formulate this optimization problem as a linear program and use a graphical technique to solve for the optimum. Also, compute the maximum profit in dollars (Note: 100 cents = 1 US dollar).

(b) The marketing department proposed that the confectioner add a third kind of candy bar named Munchies into the production line in order to try to make more profit. Munchies has a profit of 80 cents per box with the following operation times: blending time = 2 minutes, cooking time = 5 minutes, packing time = 2 minutes. Due to some adjustments in the production equipment required to accommodate the making of Munchies, the maximum hours available for the blending equipment is now reduced by 3 machine hours, the maximum for the cooking equipment is reduced by 2 machine hours, and the maximum for the packaging equipment is reduced by 2 machine hours.

As a ChEPS graduate working for the confectioner, do you agree with this proposal from marketing? Support the conclusion you make analytically (if you use linear programming to prove your answer, use the Simplex algorithm to solve the problem). There is no minimum number of boxes the confectioner is obligated to make for each kind of candy bars.

44. *Optimal Production of Two Chemicals as an LP Problem*

A chemical manufacturer plans to make two products P and Q by batch reaction. By varying reaction conditions and batch time, both products can be made from the same reactants A and B as follows:

Reaction 1: $0.5 \text{ kg } A + 0.6 \text{ kg } B \rightarrow 1 \text{ kg } P + \text{waste}$

Required batch time (including time to charge and empty reactor) = 2.5 hours

Reaction 2: $0.2 \text{ kg } A + 0.9 \text{ kg } B \rightarrow 1 \text{ kg } Q + \text{waste}$

Required batch time (including time to charge and empty reactor) = 5 hours

P can be sold at a profit of \$ 55/ton, and Q at a profit of \$ 80/ton. The company plans to use the same batch reactor to produce both products. So part of the time the reactor will be used to produce P using reaction 1, and the remaining time the reactor will be used to produce Q using reaction 2. In either case, each batch produces 0.5 ton of product P or Q. The reactor operates 330 days/year for 24 hours/day. The company is able to purchase up to 600 tons/year of A and up to 1000 tons/year of B. Additional supplies of A and B are not available.

(a) Assuming you want to maximize the total annual profit from selling P and Q, formulate this problem as a linear programming (LP) problem.

(b) How much P should you produce, and how much Q should you produce? Use the Simplex algorithm to solve for the optimum.

45. *Formulation of an LP Problem*

A Thai shrimp company with a farming area of 10 rais wishes to optimize the growing of 3 kinds of shrimps, namely tiger shrimps, prawns, and rock shrimps. The company can sell tiger shrimps at 100 baht a kilogram, prawns at 160 baht a kilogram, and rock shrimps at 200 baht a kilogram. The average yield per rai is 1,500 kilograms of tiger shrimps, 1,000 kilograms of prawns, and 800 kilograms of rock shrimps. Labor required per rai during each harvest cycle is 3 man-days, 6 man-days, and 4 man-days for tiger shrimps, prawns, and rock shrimps, respectively. No more than 50 man-days are available during the harvest cycle, and each worker is paid 300 baht per man-day. Feed and other operating expenses cost 60 baht per kilogram, while feed requirements are: 300 kilograms per rai of tiger shrimps, 450 kilograms per rai of prawns, and 500 kilograms per rai of rock shrimps.

(a) Based on this information, formulate this problem as an LP model to determine the optimal usage of the farming area such that the profit in each harvest cycle is maximized.

(b) Carry out 1 iteration of Simplex to solve this LP model. Is the solution optimal after 1 iteration?

46. *Simplex Algorithm, I*

Use the Simplex algorithm to solve for the optimum of the following LP:

$$
\begin{aligned}
&\text{Maximize} && f(\mathbf{x}) = 2x_1 - x_2 + x_3 - x_4 \\
&\text{s.t.} && \\
& && -x_1 - x_2 - x_3 + x_4 \geq -4 \\
& && -2x_1 + x_2 - x_3 \geq -12 \\
& && \\
& && x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0
\end{aligned}
$$

47. *Simplex Algorithm, II*

Use the Simplex algorithm to solve for the optimum of the following LP:

$$
\begin{aligned}
&\text{Minimize} && \boldsymbol{f}(\mathbf{x}) = x_1 + 2x_2 + 2x_3 - x_4 \\
&\text{s.t.} && \\
& && x_1 + 2x_2 - 2x_3 \geq 0 && (1) \\
& && 2x_1 - 4x_2 - x_3 + x_4 \leq 14 && (2) \\
& && -2x_1 + x_2 + 3x_3 - x_4 \geq -6 && (3) \\
& && x_1 + x_2 + x_3 + x_4 = 12 && (4) \\
& && \\
& && x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

You are <u>not</u> allowed to eliminate any variables and thus reduce the number of constraints when using Simplex. A close inspection of the LP formulation shows that one feasible basic solution can be found by making x_2 and the slack/surplus variables of Constraint (1) and Constraint (3) non-basic variables. Start your Simplex with this basic solution.

48. *Simplex Algorithm, III*

Use the Simplex algorithm to solve for the optimum of the following LP:

$$
\begin{aligned}
&\text{Maximize} && \boldsymbol{f}(\mathbf{x}) = x_1 - 2x_2 + 4x_3 \\
&\text{s.t.} && \\
& && 2x_1 + x_2 - x_3 \geq 6 \\
& && x_1 + x_2 + x_3 \leq 20 \\
& && x_2 + x_3 \geq 8 \\
& && \\
& && x_1 \geq 0,\ x_2 \text{ is unrestricted},\ x_3 \geq 0
\end{aligned}
$$

You are <u>not</u> allowed to eliminate any constraint when using Simplex.

49. *Steel Blending Problem*

The Pittsburgh Steel (PS) Co. has been contracted to produce a new type of steel which has the following tight quality requirements:

Content	At Least	Not More Than
Carbon	3.00%	3.50%
Chrome	0.30%	0.45%
Manganese	1.35%	1.65%
Silicon	2.70%	3.00%

PS has the following materials available for mixing up a batch:

	Cost/lb.	Percent Carbon	Percent Chrome	Percent Manganese	Percent Silicon	Amount Available
Pig Iron 1	0.03	4.0	0	0.9	2.25	Unlimited
Pig Iron 2	0.0645	0	10.0	4.5	15.0	Unlimited
Ferro-Silicon 1	0.065	0	0	0	45.0	Unlimited
Ferro-Silicon 2	0.061	0	0	0	42.0	Unlimited
Alloy 1	0.10	0	0	60.0	18.0	Unlimited
Alloy 2	0.13	0	20.0	9.0	30.0	Unlimited
Alloy 3	0.119	0	8.0	33.0	25.0	Unlimited
Carbide	0.08	15.0	0	0	30.0	Unlimited
Steel 1	0.021	0.4	0	0.9	0	200 lb.
Steel 2	0.02	0.1		0	0.3	0 200 lb.
Steel 3	0.0195	0.1	0	0.3	0	200 lb.

A one-ton (2000 lb.) batch must be blended which satisfies the quality requirements stated earlier. The problem now is what amounts of each of the eleven materials should be blended together so as to minimize the cost but satisfy the quality requirements. An experienced steel man claims that the least cost mix will not use more than nine of the eleven raw materials. Formulate this problem as an LP problem. To verify the steel man's claim, use LINDO to solve for the optimum. Is the steel man's claim correct or wrong?

50. ***Reclaiming Solid Wastes***

The SAVE-IT COMPANY operates a reclamation center that collects four types of solid waste materials and treats them so that they can be amalgamated into a salable product (Treating and amalgamating are separate processes.) Three different grades of this product can be made (see the first column of Table 1), depending upon the mix of the materials used. Although there is some flexibility in the mix for each grade, quality standards may specify the minimum or maximum amount allowed for the proportion of a material in the product grade. (This proportion is the weight of the material expressed as a percentage of the total weight for the product grade.) For each of the two higher grades, a fixed percentage is specified for one of the materials. These specifications are given in Table 1 along with the cost of amalgamation and the selling price for each grade.

The reclamation center collects its solid waste materials from regular sources and so is normally able to maintain a steady rate for treating them. Table 2 gives the quantities available for collection and treatment each week, as well as the cost of treatment, for each type of material.

The Save-It Co. is solely owned by Green Earth, an organization devoted to dealing with environmental issues, so Save-It's profits are used to help support Green Earth's activities. Green Earth has raised contributions and grants, amounting to $30,000 per week, to be used exclusively to cover the entire treatment cost for the solid waste materials. The board of directors of Green Earth has instructed the management of Save-It to divide this money among the materials in such a way that *at least half* of the amount available of each material is actually collected and treated. These additional restrictions are listed in Table 2.

Within the restrictions specified in Tables 1 and 2, management wants to determine the *amount* of each product grade to produce *and* the exact *mix* of materials to be used for each grade. The objective is to maximize the net weekly profit (total sales income *minus* total amalgamation cost), exclusive of the fixed treatment cost of $30,000 per week that is being covered by gifts and grants. Formulate this problem as an LP and solve it using LINDO.

Table 1: Product data for Save-It Co.

Grade	**Specification**	**Amalgamation Cost per Pound ($)**	**Selling Price per Pound ($)**
A	Material 1: Not more than 30% of total Material 2: Not less than 40% of total Material 3: Not more than 50% of total Material 4: Exactly 20% of total	3.00	8.50
	Material 1: Not more than 50% of total		

B	Material 2: Not less than 10% of total Material 4: Exactly 10% of total	2.50	7.00
C	Material 1: Not more than 70% of total	2.00	5.50

Table 2: Solid waste materials data for Save-It Co.

Material	**Pounds per Week Available**	**Treatment Cost per Pound ($)**	**Additional Restrictions**
1	3,000	3.00	1. For each material, at least half of the pounds per week available should be collected and treated.
2	2,000	6.00	
3	4,000	4.00	
4	1,000	5.00	2. $30,000 per week should be used to treat these materials.

51. *The Method of Lagrange Multipliers, I*

Use the method of Lagrange Multiplier to solve for <u>all</u> stationary points of the following constrained optimization problem:

$$f(x) = x_1^2 x_2 - x_1 x_2 + 3x_2$$

$$s.t. \quad x_1 x_2 \le 10$$

$$x_1 + x_2 = 4$$

You need <u>not</u> characterize the stationary points, but do compute the objective function value of each stationary point and their Langrage multipliers.

52. *The Method of Lagrange Multipliers, II*

Consider the following nonlinear constrained optimization problem:

$$\text{Minimize} \quad f(\boldsymbol{x}) = x_1^2 + 2x_1x_2 + (x_2 - 1)^2$$

$$\text{subject to} \quad g(\boldsymbol{x}) = x_1^2 + x_2^2 - 9 \le 0$$
$$h(\boldsymbol{x}) = x_1 + 2x_2^2 = 10$$

Use the Method of Lagrange Multipliers to find all the stationary points. For each stationary point you find, be sure to calculate the values of the Lagrange Multipliers as well as the values of $f(\boldsymbol{x})$. You do not need to classify each stationary point.

53. *The Method of Lagrange Multipliers, III*

Use the method of Lagrange Multiplier to solve for all stationary points of the following constrained optimization problem:

$$f(x) = x_1 x_2$$

$$s.t. \quad x_1 + 2x_2 \le 2$$

$$2x_1^2 - x_2 = 6$$

You need not characterize the stationary points, but do compute the objective function value of each stationary point and their Langrage multipliers.

54. *Method of Lagrange Multipliers and fmincon*

(a) Use the method of Lagrange multipliers to determine the stationary points of the following nonlinear constrained problem. There is no need to classify the calculated stationary points as maximum, minimum, or saddle point, but you must calculate the values of the Lagrange multiplier and the objective function of each stationary point.

Minimize $\quad f(x,y) = xy^2$

s.t. $\quad x^2 - y^2 \le 16$

$$x - y = 3$$

(b) Write down the MATLAB syntax (input commands and M-files) and use *fmincon* to solve the following constrained NLP problem:

Maximize $\quad f(\mathbf{x}) = 3x_1 \exp(-0.1x_1x_6) + 4x_2 + x_3^2 + 7x_4 + \dfrac{10}{x_5} + x_6$

s.t.

$x_2 x_6 = 5$ (Constraint 1)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$ (Constraint 2)

$2x_1 + x_2 + x_3 + 3x_4 \ge 2$ (Constraint 3)

$\dfrac{x_1}{x_2} + x_3^2 x_4^3 = 1$ (Constraint 4)

$-8x_1 - 3x_2 - 4x_3 + x_4 - x_5 + x_6 \ge -10$ (Constraint 5)

$-2x_1 - 6x_2 - x_3 - 3x_4 - x_6 \ge -13$ (Constraint 6)

$-x_1 - 4x_2 - 5x_3 - 2x_4 \ge -18$ (Constraint 7)

$$\sqrt{x_5} + x_6 \le 6 \qquad \text{(Constraint 8)}$$

$$-20 \le x_i \le 20 \qquad i = 1, \ldots, 6$$

Use an initial guess of $x_i = 1.0$, $i = 1, \ldots, 6$. The solution from MATLAB is $x_1 = 1.4158$, $x_2 = 1.5583$, $x_3 = -3.0346$, $x_4 = 0.2149$, $x_5 = 6.6369$, and $x_6 = 3.2087$ with $f(\boldsymbol{x}) = 24.3585$. Which inequality constraint(s) is/are active (identify by constraint number)? Be sure to write neatly and pay attention to details such as semicolons, periods, etc., since the syntax must be precise.

55. *Using fmincon in MATLAB I, and Method of Lagrange Multipliers*

(a) Use the method of Lagrange multipliers to determine the stationary points of the following nonlinear constrained problem. There is no need to classify the calculated stationary points as maximum, minimum, or saddle point, but you must calculate the values of the Lagrange multiplier and the objective function of each stationary point.

$$\text{Minimize} \qquad f(x, y) = xy^2$$

$$\text{s.t.} \qquad x^2 - y^2 \le 16$$

$$x - y = 3$$

(b) Write down the MATLAB syntax (input commands and M-files) and use *fmincon* to solve the following constrained NLP problem:

$$\text{Maximize} \quad f(\boldsymbol{x}) = 3x_1 \exp(-0.1 x_1 x_6) + 4x_2 + x_3^2 + 7x_4 + \frac{10}{x_5} + x_6$$

s.t.

$$x_2 x_6 = 5 \qquad \text{(Constraint 1)}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10 \qquad \text{(Constraint 2)}$$

$$2x_1 + x_2 + x_3 + 3x_4 \ge 2 \qquad \text{(Constraint 3)}$$

$$\frac{x_1}{x_2} + x_3^2 x_4^3 = 1 \qquad \text{(Constraint 4)}$$

$$-8x_1 - 3x_2 - 4x_3 + x_4 - x_5 + x_6 \ge -10 \qquad \text{(Constraint 5)}$$

$$-2x_1 - 6x_2 - x_3 - 3x_4 - x_6 \ge -13 \qquad \text{(Constraint 6)}$$

$$-x_1 - 4x_2 - 5x_3 - 2x_4 \ge -18 \qquad \text{(Constraint 7)}$$

$$\sqrt{x_5} + x_6 \le 6 \qquad \text{(Constraint 8)}$$

$$-20 \le x_i \le 20 \qquad i = 1, \ldots, 6$$

Use an initial guess of $x_i = 1.0$, $i = 1, \ldots, 6$. The solution from MATLAB is $x_1 = 1.4158$, $x_2 = 1.5583$, $x_3 = -3.0346$, $x_4 = 0.2149$, $x_5 = 6.6369$, and $x_6 = 3.2087$ with $f(\boldsymbol{x}) =$

24.3585. Which inequality constraint(s) is/are active (identify by constraint number)? Be sure to write neatly and pay attention to details such as semicolons, periods, etc., since the syntax must be precise.

56. *Using fmincon to Solve 3 NLPs*

(a) Westerberg and Shah (1978) in Swaney (1990) studied the following NLP problem:

$$\begin{array}{ll} \text{Min} & 35x_1^{0.6} + 35x_2^{0.6} \\ \text{s.t.} & 600x_1 - 50x_3 - x_1x_3 + 5000 = 0 \\ & 600x_2 + 50x_3 - 15000 = 0 \\ & (0, 0, 100) \leq \boldsymbol{x} \leq (34, 17, 300) \end{array}$$

in which one local optimum ($f(\boldsymbol{x})$ = 286.943) and the global optimum ($f(\boldsymbol{x})$ = 189.311627) were reported by Ryoo and Sahinidis (1995). Use *fmincon* to find both solutions.

(b) Stoecker (1971) in Liebman *et al.* (1986) studied the following insulated steel tank design NLP problem:

$$\begin{array}{ll} \text{Min} & 400x_1^{0.9} + 1000 + 22(x_2 - 14.7)^{1.2} + x_4 \\ \text{s.t.} & x_2 = \exp[-3950/(x_3 + 460) + 11.86] \\ & 144(80 - x_3) = x_1x_4 \\ & (0, 14.7, -459.67, 0) \leq \boldsymbol{x} \leq (15.1, 94.2, 80, \infty) \end{array}$$

in which the global optimum ($f(\boldsymbol{x})$ = 5194.866243) was reported by Ryoo and Sahinidis (1995). Use *fmincon* to find the global optimum.

(c) Stephanopoulos and Westerberg (1975) studied the following NLP problem of a design of three-stage process system with recycle.

$$\begin{array}{ll} \text{Min} & x_1^{0.6} + x_2^{0.6} + x_3^{0.4} - 4x_3 + 2x_4 + 5x_5 - x_6 \\ & \\ \text{s.t.} & -3x_1 + x_2 - 3x_4 = 0 \\ & -2x_2 + x_3 - 2x_5 = 0 \\ & 4x_4 - x_6 = 0 \\ & x_1 + 2x_4 \leq 4 \\ & x_2 + x_5 \leq 4 \\ & x_3 + x_6 \leq 6 \\ & (0, 0, 0, 0, 0, 0) \leq \boldsymbol{x} \leq (3, 4, 4, 2, 2, 6) \end{array}$$

in which the global optimum ($f(\boldsymbol{x})$ = –13.401904) was reported by Ryoo and Sahinidis (1995). Use both the interior-reflective Newton method and the SQP method in *fmincon* to find the global optimum.

<u>Hint</u>: You may need to try different starting points to find the solutions.

57. *Using fmincon in MATLAB, II*

Write down the MATLAB syntax (input commands and M-files) using *fmincon* to solve the following constrained NLP problem:

$$\text{Maximize} \quad f(\boldsymbol{x}) = x_3 \sin(x_1 x_2) - x_3 x_4 + x_4^3 + \frac{4}{x_1} - 2\ln(x_1 x_4)$$

s.t.

$$2x_1 - 2x_2 + x_3 - 4x_4 = -10 \qquad \text{(Constraint 1)}$$

$$\sqrt{2x_1} - \frac{x_3}{x_2} + \exp(-x_4) \geq 2 \qquad \text{(Constraint 2)}$$

$$x_1 + 2x_2 - x_4 \geq 0 \qquad \text{(Constraint 3)}$$

$$\frac{x_1}{x_2} + 8.28 \times 10^{-4} x_3 + x_4 = 3 \qquad \text{(Constraint 4)}$$

$$2x_1 + 8x_2 + x_3 + 4x_4 \leq 100 \qquad \text{(Constraint 5)}$$

$$x_2 + 0.0693 \exp\left(\frac{x_2 x_4}{x_3}\right) = -14 \qquad \text{(Constraint 6)}$$

$$x_1 x_2 x_4 - 30x_3 \leq 40 \qquad \text{(Constraint 7)}$$

$$3x_1 + x_3 + 6x_4 = 50 \qquad \text{(Constraint 8)}$$

$$-100 \leq x_i \leq 100 \qquad i = 1, \ldots, 4$$

Use an initial guess of $x_i = 2$, $i = 1, \ldots, 4$. The solution from MATLAB is $x_1 = 33.8022$, $x_2 = -14.1737$, $x_3 = -84.1338$, and $x_4 = 5.4545$ with $f(\boldsymbol{x}) = 695.0049$. Which inequality constraint(s) is/are active (identify by constraint number)? Be sure to write neatly and pay attention to details such as semicolons, periods, etc., since the syntax must be precise.

58. *Formulation and Solving an ILP Problem*

Three high-value chemical products, *A*, *B*, and *C* are to be produced from two raw materials, namely RawMat 1 and RawMat 2. The production of 1 batch of *A* requires 4 kg of RawMat 1 and 2.5 kg RawMat 2 and 3.5 hours of labor. On the other hand, the production of 1 batch of *B* requires 1 kg of RawMat 1 and 3.5 kg of RawMat 2 and 2 hours of labor. Finally, the production of 1 batch of *C* requires 8.5 kg of RawMat 1 and 4 kg of RawMat 2 and 5 hours of labor.

In terms of resource capacities for one week of production, 1,000 kg of RawMat 1 and 1,200 kg of RawMat 2 are available, while 20 employees each working 40 hours are used.

The following marketing data are available:

	Profit/batch ($)	Demand/week
Product *A*	10.00	40 batches
Product *B*	5.00	100 batches
Product *C*	15.00	30 batches

(a) Formulate this optimization problem in which the goal is to determine the number of batches products *A*, *B*, and *C* are to be produced in a week so as to maximize the total profit. Also, classify the problem. Note that a partial batch cannot be produced.

(b) Solve the ILP problem using LINDO.

59. *Integer Linear Programming*

Solve the following problems via the branch and bound method. You may use LINDO to solve each LP sub-problem (node).

(a) Maximize $f(\mathbf{x}) = 75x_1 + 6x_2 + 3x_3 + 33x_4$

Subject to
$$774x_1 + 76x_2 + 22x_3 + 42x_4 \leq 875$$
$$67x_1 + 27x_2 + 794x_3 + 53x_4 \leq 875$$
x_1, x_2, x_3, x_4 either 0 or 1

(b) Maximize $f(\mathbf{x}) = 2x_1 + x_2$

Subject to
$$x_1 + x_2 \leq 5$$
$$x_1 - x_2 \geq 0$$
$$6x_1 + 2x_2 \leq 21$$
$x_1, x_2 \geq 0$ and integer

60. *Using LINDO to Solve Scheduling Problems*

(a) Recall the 10-product, 4-unit single-stage scheduling problem in Problem #12. Use LINDO to solve for the optimal schedule with the shortest makespan.

(b) Consider the following 8-product, 3-unit multistage scheduling problem with unlimited intermediate storage:

	Processing Times (in minutes)							
Product	1	2	3	4	5	6	7	8
Unit 1	20	12	30	45	60	20	15	25
Unit 2	55	30	50	40	30	20	25	32
Unit 3	15	20	38	25	51	14	26	33

Use LINDO to solve for the optimal schedule with the shortest makespan.

61. *Toll-Way Staffing Problem*

Consider Problem 12 again. The Northeast Tollway out of Chicago has a toll plaza with the following staffing demands during each 24-hour period:

Hours	Collectors Needed
12 a.m. to 6 a.m.	2
6 a.m. to 10 a.m.	8
10 a.m. to 12 a.m.	4
12 a.m. to 4 p.m.	3
4 p.m. to 6 p.m.	6
6 p.m. to 10 p.m.	5
10 p.m. to 12 Midnight	3

Each collector works 4 hours, is off one hour, and then works another 4-hours. A collector can be started at any hour. Assuming the objective is to minimize the number of collectors hired, formulate this problem as an ILP problem (since the number of collectors hired cannot be fractional). Use LINDO to solve for the optimum.

62. *Completion Time Calculations and UIS Makespan Minimization*

Consider a 4-stage multiproduct plant with 4 products. The processing time matrix (in minutes) is given below.

	Products			
Units	1	2	3	4
1	5	1	2	2
2	10	3	18	2
3	10	3	10	25
4	32	10	5	15

(a) Assuming the production sequence to be 1-2-3-4 on all four processing units, use Gantt charts to calculate a complete schedule and the makespan for the following cases:

i) UIS policy
ii) NIS policy
iii) ZW policy

iv) MIS policy with ZW between units 1 and 2, NIS between units 2-3, and FIS with one storage unit between units 3 and 4.

(b) For the UIS policy, formulate the problem as an MILP, and use LINDO to solve for an optimal schedule that minimizes the makespan.

63. *UIS Scheduling with Makespan Minimization*

Consider the makespan minimization of a 5-product, 3-unit UIS multiproduct plant with unlimited intermediate storage and the following processing time matrix (in minutes):

Units	Products 1	2	3	4	5
1	15	10	10	5	8
2	8	20	15	15	14
3	5	12	16	9	18

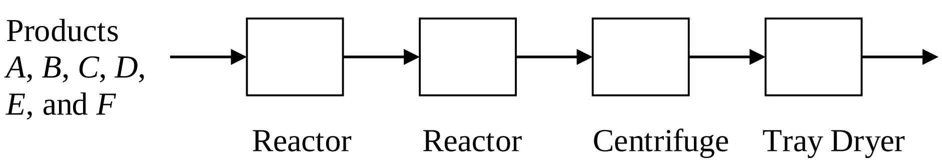
(a) Write a MATLAB code to use brute force (i.e. examine every possible solution) to find a schedule that minimizes the makespan.

(b) Formulate the problem as an MILP, and use LINDO to solve for an optimal schedule in Part (a)

64. *Optimal Scheduling of Products in a Serial Batch Process*

An important optimization problem that arises in a batch chemical plant (or non-continuous processes) is the scheduling problem. Short-term scheduling involves sequencing and scheduling the production of *N* products across *M* processing units to optimize a suitable performance or cost-based criterion (Ku *et al.* 1987).

Consider the following serial batch process which consists of 2 batch reactors followed by a centrifuge and a tray dryer. The process is to



manufacture 6 products, namely *A*, *B*, *C*, *D*, *E*, and *F*. All products must follow the same production sequence, i.e. they must first pass through the first reactor, followed by the second, then onto the centrifuge and finally the tray dryer. The processing time of each product required in each processing unit is given in the table below. Each product also has a delivery due date after which a penalty would be incurred.

An effective heuristic (method yielding an approximate and not necessarily optimal solution) called Duedate-Over-Penalty or DOP algorithm [Ku, H.M., and Karimi, I.A., "Scheduling in Serial Multiproduct Batch Processes with Due Date Penalties", **Ind. Eng. Chem. Res.**, **29**, 4, 580, (1990)] has been proposed to solve the due-date scheduling problem as follows:

1. Compute d_i/p_i, $i = 1, N$.

2. Sequence products in the ascending order of d_i/p_i.

3. Compute the completion times and the total penalty cost P for the above sequence.

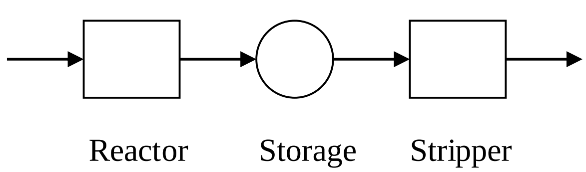
Solve the problem using the DOP algorithm, and compare the solution you obtain with the brute force approach (i.e. examining every possible solution) using MATLAB. Assume that there is unlimited intermediate storage (UIS policy) available to store intermediate products between every 2 stages.

Batch Unit	**Processing Times (Hours)** *A*	*B*	*C*	*D*	*E*	*F*
Reactor 1	5	35	10	40	25	45
Reactor 2	15	20	30	18	32	7
Centrifuge	10	15	30	12	16	22
Tray Dryer	8	10	20	10	28	38
Due Date, d_i (Days)	1.5	4.0	2.5	5.0	3.0	6.0
Penalty, p_i ($/Hour)	500	400	600	800	300	200

65. *Optimal Scheduling of Products in a Serial Batch Process with FIS Policy*

Now, consider another batch process with only 2 processing stages as shown:

Products $P1$, $P2$, $P3$, $P4$, and $P5$



This 2-unit serial system operates under the Finite Intermediate Storage policy (FIS) with one storage vessel, and is used to produce 5 products, namely $P1$, $P2$, $P3$, $P4$, and $P5$. The processing time matrix for this system is given below.

Batch Unit	Processing Times (Minutes) $P1$	$P2$	$P3$	$P4$	$P5$
Reactor	10	6	2	4	15
Stripper	9	5	14	8	20

For the FIS policy, the following recurrence relations were developed by Ku (1984) to compute the completion time of each product on each processing unit for a given production sequence: $k_1 - k_2 - k_3 - \ldots - k_N$ where product k_i is in the ith position in the production sequence:

$$C_{ij} = \text{Max}\left[\, C_{(i-1)j}\,,\, C_{i(j-1)}\,,\, C_{(i-1-z\,)(j+1)} - t_{k\,j} \,\right] + t_{k\,j} \qquad i = 1, \ldots, N \quad j = 1, \ldots, M$$

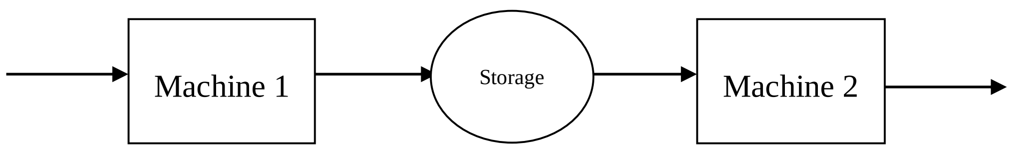
$$C_{ij} = 0 \ \text{ if } \ i \leq 0 \ \text{ or } \ j \leq 0$$

where C_{ij} = completion time of the ith product on the jth processing unit
= the time at which the ith product leaves unit j
$t_{k\,j}$ = processing time of product k_i on unit j
z_j = number of storage units available in the FIS policy between unit j and unit $(j+1)$

Formulate this FIS scheduling problem as an MILP model and determine an optimal production sequence that minimizes the makespan in the production of the 5 products. Also, use MATLAB to implement the brute force approach by examining every possible solution in order to find the same optimal schedule solved in the MILP model.

66. *A Two-Machine Scheduling Problem under UIS and FIS Storage Policies*

Consider a two-machine scheduling problem of 6 products as shown with the following processing time matrix:



		Processing Times t_{ij} Machine 1	2
Product	A	50	10
	B	10	15
	C	15	30
	D	26	5
	E	6	20
	F	8	26

(a) A simple optimization algorithm called Johnson's Rule (March, 1954) exists that gives an optimal sequence with the minimum makespan when there is unlimited intermediate storage (UIS operating policy). The Rule can be stated as follows:

1. Create a list *P* containing products whose $t_{i1} \le t_{i2}$ and are arranged in the increasing value of their t_{i1}.
2. Create a second list *Q* containing products whose $t_{i1} > t_{i2}$ and are arranged in the decreasing value of their t_{i2}.
3. Concatenate *P* and *Q*, i.e. combine them so that the optimal sequence is $P \cup Q$.

Use Johnson's Rule to determine the sequence with the minimum makespan in this scheduling problem. Also, calculate this minimum makespan.

(b) Write out the complete formulation for this UIS scheduling problem. In your formulation, you must first define all the variables, and then state the objective function and all the constraints term by term. Do not use any summation sign to show your objective function or constraints. Processing times should also be written down as numbers, not as symbols, in the formulation.

(c) When the storage operating policy is FIS (finite intermediate storage), the completion time of a product in position *i* from machine *j* is given as (Ku and Karimi, 1988):

$$C_{ij} = \max(C_{(i-1)j}, C_{i(j-1)}, C_{(i-z-1)(j+1)} - t_{ij}) + t_{ij}$$

where z = the number of storage vessels, e.g $z = 0$ if there is no storage and $z = \infty$ if there is unlimited intermediate storage. Using the optimal sequence under UIS obtained in Part (a), calculate its makespan when $z = 1$. Note that this sequence is not necessarily an optimal one under the FIS policy.